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Multilinear variants of Pietsch's composition theorem

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ABSTRACT

Various concepts of multilinear summing operators were introduced in the last years, by extending the well-known one from the linear case. In this paper, we prove that, as in the linear case, there is a splitting theorem for dominated operators. As a consequence of this result, we prove various multilinear variants of Pietsch's composition theorem.

1. Introduction and notation

In the theory of linear summing operators, Pietsch's composition theorem, which asserts that if $p, q \in (1, \infty)$ and $r \in [1, \infty)$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\pi_q \circ \pi_p \subset \pi_r$, is one of the fundamental results, see [5,6,14,15,17,19]. In this context, question is if there is an extension of Pietsch's composition theorem to the multilinear settings. Since the proof of Pietsch's composition theorem is based on the Grothendieck–Pietsch domination theorem, it seems natural to work in the multilinear case with the class for which there is a domination theorem. One such a class is the class of dominated operators, see [4,8,10,12,13]. In this paper we prove that, as in the linear case, the class of dominated operators has some general splitting theorem and, as a consequence, we deduce some possible extensions of Pietsch's composition theorem to multilinear settings. We fix some notations and notions used through the paper.

Let X be a Banach space, B_X the closed unit ball of X and X^* the dual of X. For $0 < p < \infty$ and every $x_1, \ldots, x_n \in X$ we denote

$$w_p(x_i \mid 1 \leqslant i \leqslant n) = \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \left| x^*(x_i) \right|^p \right)^{\frac{1}{p}}, \qquad l_p(x_i \mid 1 \leqslant i \leqslant n) = \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

Definition 1. Let $p \in [1, \infty)$ and X, Y be Banach spaces. A linear continuous operator $U : X \to Y$ is called p-summing if and only if there exists $C > 0$ such that for each $(x_i)_{1 \leqslant i \leqslant m} \subset X$ the following hold

$$\left(\sum_{i=1}^{m} \left\| U(x_i) \right\|^p \right)^{\frac{1}{p}} \leqslant C w_p(x_i \mid 1 \leqslant i \leqslant m) \tag{$*$}$$

and the p-summing norm of U is

$$\pi_p(U) = \inf\left\{ C \mid C \text{ verifies } (*) \right\}.$$

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We denote by $\pi_p(X,Y)$ the class of all p-summing operators from X into Y; see [5,6,14,15,17,19].

In the multilinear case there is a general extension of the linear summing operators, see [1,2,10–12]. In this paper, to avoid unpleasant repetitions, all $(n-)$ multilinear continuous operators are always defined on a Cartesian product of $(n-)$ Banach spaces with values in a Banach space.

Definition 2. Let $p_1,\dots,p_n \in [1,\infty)$ and $t \in (0,\infty)$ be such that $\frac{1}{t} \leqslant \frac{1}{p_1}+\dots+\frac{1}{p_n}$. A multilinear continuous operator $U : X_1 \times \dots \times X_n \to Y$ is called $(t;p_1,\dots,p_n)$-summing if and only if there exists $C>0$ such that for each $(x_i^j)_{1\leqslant i\leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ the following hold

$$\left(\sum_{i=1}^m \left\|U\left(x_i^1,\dots,x_i^n\right)\right\|^t\right)^{\frac{1}{t}} \leqslant C w_{p_1}\left(x_i^1 \;\middle|\; 1\leqslant i\leqslant m\right)\cdots w_{p_n}\left(x_i^n \;\middle|\; 1\leqslant i\leqslant m\right) \tag{$**$}$$

and

$$\pi_{t;p_1,\dots,p_n}(U) = \inf\left\{C \;\middle|\; C \text{ verifies } (**)\right\}.$$

We denote by $\pi_{t;p_1,\dots,p_n}(X_1,\dots,X_n;Y)$ the class of all $(t;p_1,\dots,p_n)$-summing operators from $X_1\times\dots\times X_n$ into Y on which $\pi_{t;p_1,\dots,p_n}$ is a norm if $t \geqslant 1$ (t-norm, if $t<1$). When $p_1=\dots=p_n=p$, and in this case $\frac{1}{t}\leqslant\frac{n}{p}$, we write simply $\pi_{t;p}$ instead of $\pi_{t;p_1,\dots,p_n}$. In the linear case i.e. $n=1$ we get the well-known definition of (t,p)-summing operators.

There are two particular cases of this general definition which, to avoid some misunderstandings, we state explicitly in the sequel. The first is the class of dominated operators, i.e. those for which $\frac{1}{t}=\frac{1}{p_1}+\dots+\frac{1}{p_n}$.

Definition 3. Let $p_1,\dots,p_n \in [1,\infty)$ and define $t\in(0,\infty)$ by $\frac{1}{t}=\frac{1}{p_1}+\dots+\frac{1}{p_n}$. A multilinear continuous operator $U : X_1\times\dots\times X_n \to Y$ is called $(p_1,\dots,p_n)$-dominated if and only if there exists $C>0$ such that for each $(x_i^j)_{1\leqslant i\leqslant m}\subset X_j$ $(1\leqslant j\leqslant n)$ the following hold

$$\left(\sum_{i=1}^m \left\|U\left(x_i^1,\dots,x_i^n\right)\right\|^t\right)^{\frac{1}{t}} \leqslant C w_{p_1}\left(x_i^1 \;\middle|\; 1\leqslant i\leqslant m\right)\cdots w_{p_n}\left(x_i^n \;\middle|\; 1\leqslant i\leqslant m\right) \tag{$***$}$$

and

$$\delta_{p_1,\dots,p_n}(U) = \inf\left\{C \;\middle|\; C \text{ verifies } (***)\right\}.$$

We denote by $\delta_{p_1,\dots,p_n}(X_1,\dots,X_n;Y)$ the class of all $(p_1,\dots,p_n)$-dominated operators from $X_1\times\dots\times X_n$ into Y on which $\delta_{p_1,\dots,p_n}$ is a norm if $t\geqslant 1$ (t-norm, if $t<1$). Dominated $(p,\dots,p)$-operators, in which case we have $t=\frac{p}{n}$, are called p-dominated and we write simply δ_p instead of $\delta_{p,\dots,p}$. Observe that p-dominated operators are exactly $(\frac{p}{n};p,\dots,p)$-summing operators.

The main feature of this class is that we have a Grothendieck–Pietsch domination theorem, see [8,10,12,16].

Domination theorem. *Let $\Omega_j \subseteq B_{X_j^*}$ be a weak*-compact and norming subset of $B_{X_j^*}$ and $p_1,\dots,p_n\in[1,\infty)$. A multilinear continuous operator $U : X_1\times\dots\times X_n\to Y$ is $(p_1,\dots,p_n)$-dominated if and only if there exists $K>0$ and regular Borel probability measures μ_j on Ω_j $(1\leqslant j\leqslant n)$ such that for each $(x_1,\dots,x_n)\in X_1\times\dots\times X_n$ we have*

$$\left\|U(x_1,\dots,x_n)\right\| \leqslant K\left(\int_{\Omega_1}\left|x_1^*(x_1)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}}\cdots\left(\int_{\Omega_n}\left|x_n^*(x_n)\right|^{p_n} d\mu_n\left(x_n^*\right)\right)^{\frac{1}{p_n}}. \tag{$****$}$$

Moreover $\delta_{p_1,\dots,p_n}(U)=\inf\{K \mid K \text{ verifies } (****)\}$; see [8,10,12,16].

From domination theorem we deduce:

Inclusion theorem. *Let $p_1,\dots,p_n,\ s_1,\dots,s_n\in[1,\infty)$ be such that $p_1\leqslant s_1,\dots,p_n\leqslant s_n$. Then $\delta_{p_1,\dots,p_n}(X_1,\dots,X_n;Y)\subset\delta_{s_1,\dots,s_n}(X_1,\dots,X_n;Y)$.*

A second is the class of p-summing operators, i.e. those in which $t=p_1=\dots=p_n=p\geqslant 1$.

Definition 4. Let $1 \leqslant p < \infty$. A multilinear continuous operator $U : X_1 \times \cdots \times X_n \to Y$ is called p-summing, if there exists a constant $C \geqslant 0$ such that for each $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ the following hold

$$\left(\sum_{i=1}^{m} \left\| U\left(x_i^1, \ldots, x_i^n\right) \right\|^p \right)^{\frac{1}{p}} \leqslant C w_p\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_p\left(x_i^n \mid 1 \leqslant i \leqslant m\right) \qquad (*****)$$

and the p-summing norm of U is $\pi_p(U) = \inf\{C \mid C \text{ verifies } (*****)\}$.

In all the above definitions, a more precise notation is $\pi^n_{t;p_1,\ldots,p_n}$, $\delta^n_{p_1,\ldots,p_n}$, δ^n_p, π^n_p instead of $\pi_{t;p_1,\ldots,p_n}$, $\delta_{p_1,\ldots,p_n}$, δ_p, π_p. However, when this will cause no confusion, we prefer to use these simple notations, the distinction between linear and multilinear notations will be clear from the context.

All these three classes of multilinear operators verify the axioms of a λ-Banach ideal (for $0 < \lambda \leqslant 1$) (of n-linear operators) as this notion was introduced by A. Pietsch in [16], see also [7] which we recall now. For a natural number n, Banach spaces $X_1, \ldots, X_n$, Y we denote $L(X_1, \ldots, X_n; Y)$ the Banach space of all n-linear continuous operators, which we call simply multilinear continuous, when the natural number n will be clear from the context.

Definition 5. A subclass $\mathcal{A}$ of the class $\mathcal{L}$ of all n-linear continuous operators between Banach spaces is called an ideal if

$(M1)$ For all Banach spaces $X_1, \ldots, X_n$, Y the component $\mathcal{A}(X_1, \ldots, X_n; Y) \stackrel{\text{def}}{=} L(X_1, \ldots, X_n; Y) \cap \mathcal{A}$ is a linear subspace of $L(X_1, \ldots, X_n; Y)$.

$(M2)$ If

$$X_1 \xrightarrow{A_1} Y_1, \ldots, X_n \xrightarrow{A_n} Y_n, \qquad Y_1 \times \cdots \times Y_n \xrightarrow{T} Z \xrightarrow{S} W$$

where all A_j and S are bounded linear, $T \in \mathcal{A}(Y_1, \ldots, Y_n; Z)$, then the composition $S \circ T \circ (A_1, \ldots, A_n) \in \mathcal{A}(X_1, \ldots, X_n; W)$.

Above $T \circ (A_1, \ldots, A_n) : X_1 \times \cdots \times X_n \to Z$ is defined by

$$T \circ (A_1, \ldots, A_n)(x_1, \ldots, x_n,) = T\left(A_1(x_1), \ldots, A_n(x_n)\right).$$

$(M3)$ $[\mathbb{K}^n \ni (\lambda_1, \ldots, \lambda_n) \to \lambda_1 \cdots \lambda_n \in \mathbb{K}] \in \mathcal{A}$.

A (quasi-)normed ideal is a pair $(\mathcal{A}, \| \ \|_{\mathcal{A}})$, where $\mathcal{A}$ is an ideal and $\| \ \|_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ is an ideal (quasi-)norm, i.e.

$(M1')$ $\| \ \|_{\mathcal{A}}$ restricted to each component is a (quasi-)norm.

$(M2')$ $\|S \circ T \circ (A_1, \ldots, A_n)\|_{\mathcal{A}} \leqslant \|S\| \mathcal{A}(T) \|A_1\| \cdots \|A_n\|$ in the situation of $(M2)$.

$(M3')$ $\|[\mathbb{K}^n \ni (\lambda_1, \ldots, \lambda_n) \to \lambda_1 \cdots \lambda_n \in \mathbb{K}]\|_{\mathcal{A}} = 1$.

The terms λ-normed (for $0 < \lambda \leqslant 1$), normed, quasi-Banach, λ-Banach ideal and Banach ideal are used in the obvious way.

In [13], if $\mathcal{A}$ and $\mathcal{B}$ are two λ-Banach n-ideals, $1 \leqslant q < \infty$, it is denoted by $\mathcal{A} \circ \pi_q \subset \mathcal{B}$ the fact that, if $A_1 \in \pi_q(X_1, Y_1)$, $\ldots, A_n \in \pi_q(X_n, Y_n)$ and $T \in \mathcal{A}(Y_1, \ldots, Y_n; Z)$, then $T \circ (A_1, \ldots, A_n) \in \mathcal{B}(X_1, \ldots, X_n; Z)$.

In this paper we will be interested in the reverse problem: If $\mathcal{A}$ and $\mathcal{B}$ are two λ-Banach n-ideals we denote by $\pi_q \circ \mathcal{A} \subset \mathcal{B}$ the fact that, if $U \in \mathcal{A}(X_1, \ldots, X_n; Y)$ and $T \in \pi_q(Y, Z)$ then $T \circ U \in \mathcal{B}(X_1, \ldots, X_n; Z)$.

To avoid some possible confusions we denote by $(\rho_n)_{n \in \mathbb{N}}$ the sequence of Rademacher functions. For $1 \leqslant p < \infty$ we denote by p^* for the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$; if $p = 1$ we take $p^* = \infty$ and consider $\frac{1}{\infty} = 0$.

In this paper all notation and terminology, not otherwise explained, are as in [5] or [6].

2. The splitting results

We prove first that one of possible multilinear extensions of Pietsch's composition theorem is not true in case $n \geqslant 2$. The proof was suggested by the linear case, see [5, Ex. 11.23]. We use the usual notation that if $a = (a_n)_{n \in \mathbb{N}}$, $b = (b_n)_{n \in \mathbb{N}}$ are two scalar sequences by ab we denote their pointwise multiplication i.e. $ab = (a_n b_n)_{n \in \mathbb{N}}$. In the same way, if $a_1, \ldots, a_n$ are scalar sequences, we denote their pointwise multiplication by $a_1 \cdots a_n$.

Proposition 1.

(i) *Let n be a natural number, $a \in l_\infty$, $M_a : \underbrace{c_0 \times \cdots \times c_0}_{n \text{ times}} \to c_0$ the multiplication operator, $M_a(x_1, \ldots, x_n) = a x_1 \cdots x_n$. Let $p_1, \ldots, p_n \in [1, \infty)$ and $t \in (0, \infty)$ be such that $\frac{1}{t} \leqslant \frac{1}{p_1} + \cdots + \frac{1}{p_n}$. Then M_a is $(t; p_1, \ldots, p_n)$-summing if and only if the series $\sum_{k=1}^{\infty} |a_k|^t$ is convergent. Further $\pi_{t;p_1,\ldots,p_n}(M_a) = (\sum_{k=1}^{\infty} |a_k|^t)^{\frac{1}{t}}$.*

(ii) *Let n be a natural number, $p, q \in (1, \infty)$, $r \in [1, \infty)$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $\pi_q \circ \delta_p^n \subset \delta_r^n$ if and only if $n = 1$.*

Proof. (i) Suppose M_a is $(t; p_1, \ldots, p_n)$-summing. Since for $s \geqslant 1$, $w_s(e_i \mid 1 \leqslant i \leqslant m) = \|I : l_1 \hookrightarrow l_s\| = 1$, for each $m \in \mathbb{N}$, we have $(\sum_{i=1}^{m} \|M_a(e_i, \ldots, e_i)\|^t)^{\frac{1}{t}} \leqslant \pi_{t;p_1,\ldots,p_n}(M_a)$.

From $M_a(e_i, \ldots, e_i) = a_i e_i$ we get $(\sum_{i=1}^{m} |a_i|^t)^{\frac{1}{t}} \leqslant \pi_{t;p_1,\ldots,p_n}(M_a)$, hence the series $\sum_{k=1}^{\infty} |a_k|^t$ is convergent and $(\sum_{k=1}^{\infty} |a_k|^t)^{\frac{1}{t}} \leqslant \pi_{t;p_1,\ldots,p_n}(M_a)$.

Conversely, suppose that the series $\sum_{k=1}^{\infty} |a_k|^t$ is convergent. For each $(x_i^1, \ldots, x_i^n) \in c_0 \times \cdots \times c_0$ $(1 \leqslant i \leqslant m)$ we have

$$\left\| M_a\left(x_i^1, \ldots, x_i^n\right) \right\|^t \leqslant \sum_{k=1}^{\infty} |a_k|^t \left|\left\langle x_i^1, e_k \right\rangle\right|^t \cdots \left|\left\langle x_i^n, e_k \right\rangle\right|^t. \tag{1}$$

Since $\frac{1}{t} \leqslant \frac{1}{p_1} + \cdots + \frac{1}{p_n}$, using Holder's inequality we get

$$\left(\sum_{i=1}^{m} \left|a_i^1\right|^t \cdots \left|a_i^n\right|^t \right)^{\frac{1}{t}} \leqslant \left(\sum_{i=1}^{m} \left|a_i^1\right|^{p_1} \right)^{\frac{1}{p_1}} \cdots \left(\sum_{i=1}^{m} \left|a_i^n\right|^{p_n} \right)^{\frac{1}{p_n}} \quad \text{for each scalars } a_i^j. \tag{2}$$

Using (2) from (1) we deduce

$$\begin{aligned} \sum_{i=1}^{m} \left\| M_a\left(x_i^1, \ldots, x_i^n\right) \right\|^t &\leqslant \sum_{k=1}^{\infty} |a_k|^t \left(\sum_{i=1}^{m} \left|\left\langle x_i^1, e_k \right\rangle\right|^t \cdots \left|\left\langle x_i^n, e_k \right\rangle\right|^t \right) \\ &\leqslant \sum_{k=1}^{\infty} |a_k|^t \left(\sum_{i=1}^{m} \left|\left\langle x_i^1, e_k \right\rangle\right|^{p_1} \right)^{\frac{t}{p_1}} \cdots \left(\sum_{i=1}^{m} \left|\left\langle x_i^n, e_k \right\rangle\right|^{p_n} \right)^{\frac{t}{p_n}} \\ &\leqslant \sum_{k=1}^{\infty} |a_k|^t \left[w_{p_1}\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_{p_n}\left(x_i^n \mid 1 \leqslant i \leqslant m\right) \right]^t \end{aligned}$$

thus

$$\left(\sum_{i=1}^{m} \left\| M_a\left(x_i^1, \ldots, x_i^n\right) \right\|^t \right)^{\frac{1}{t}} \leqslant \left(\sum_{k=1}^{\infty} |a_k|^t \right)^{\frac{1}{t}} w_{p_1}\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_{p_n}\left(x_i^n \mid 1 \leqslant i \leqslant m\right),$$

i.e. M_a is $(t; p_1, \ldots, p_n)$-summing and $\pi_{t;p_1,\ldots,p_n}(M_a) \leqslant (\sum_{k=1}^{\infty} |a_k|^t)^{\frac{1}{t}}$.

(ii) Suppose $\pi_q \circ \delta_p^n \subset \delta_r^n$. Let $b = (b_k)_{k \in \mathbb{N}}$ be such that the series $\sum_{k=1}^{\infty} |b_k|^{\frac{p}{n}}$ is convergent and $a = (a_k)_{k \in \mathbb{N}} \in l_q$. In the diagram $\underbrace{c_0 \times \cdots \times c_0}_{n \text{ times}} \xrightarrow{M_b} c_0 \xrightarrow{M_a} c_0$, by (i), M_b is p-dominated and M_a is q-summing. Then, by hypothesis, $M_{ab} = M_a \circ M_b \in \delta_r^n$, which, again by (i), gives that the series $\sum_{k=1}^{\infty} |a_k b_k|^{\frac{r}{n}}$ is convergent.

In particular, for each $\alpha > \frac{n}{p}$, $\beta > \frac{1}{q}$, taking $a = (\frac{1}{k^\alpha})_{k \in \mathbb{N}}$, $b = (\frac{1}{k^\beta})_{k \in \mathbb{N}}$ we obtain that the series $\sum_{k=1}^{\infty} \frac{1}{k^{(\alpha+\beta)\frac{r}{n}}}$ is convergent, thus $\alpha + \beta > \frac{n}{r}$. Now we are in the situation:

$$\alpha + \beta > \frac{n}{r}, \quad \forall \alpha > \frac{n}{p} \text{ and } \forall \beta > \frac{1}{q}. \tag{$*$}$$

Passing to the limit in $(*)$ for $\alpha \to \frac{n}{p}$, $\alpha > \frac{n}{p}$ and $\beta \to \frac{1}{q}$, $\beta > \frac{1}{q}$ we obtain $\frac{n}{p} + \frac{1}{q} \geqslant \frac{n}{r}$, i.e. $n(\frac{1}{r} - \frac{1}{p}) \leqslant \frac{1}{q}$, or, if we use $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we get $\frac{n}{q} \leqslant \frac{1}{q}$ i.e. $n = 1$.

The converse is the Pietsch composition theorem. $\square$

In view of Proposition 1, it seems that an extension of Pietsch's composition theorem to the multilinear case could be the following one.

Question. Let $p, q \in (1,\infty)$, $r \in [1,\infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For what natural numbers n the inclusion $\pi_q \circ \delta_p^n \subset \pi_r^n$ is true? In other words, for what natural numbers n, if $U \in \delta_p^n(X_1,\ldots,X_n;Y)$ and $T \in \pi_q(Y,Z)$, then $T \circ U \in \pi_r^n(X_1,\ldots,X_n;Z)$?

We will prove, see Theorem 4 and Corollary 19, that for $r \in [1,2]$, the answer to this question is **Yes** for all natural numbers n. Unfortunately we do not know the answer to this question in case $r \in (2,\infty)$.

The next lemma was proved in [1, Proof of Theorem 3.10] in "the discrete case", see also [9, Proof of Theorem 5.2], in this form, in the bilinear case in [18, Lemma 2.22] and further it is explicitly stated and used in the proof of [3, Theorem 4.1]. We omit its proof.

Lemma 2. *Let $n \geqslant 2$ be a natural number, $X_1,\ldots,X_n$, Y Banach spaces and $U : X_1 \times \cdots \times X_n \to Y$ a multilinear continuous operator. Then for each $(x_i^j)_{1\leqslant i\leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ we have the equality*

$$\sum_{i=1}^m U\left(x_i^1,\ldots,x_i^n\right) = \int_{[0,1]^{n-1}}\!\!\cdots\!\int U\left(\sum_{i=1}^m x_i^1\rho_i(t_1),\ldots,\sum_{i=1}^m x_i^{n-1}\rho_i(t_{n-1}),\sum_{i=1}^m x_i^n\rho_i(t_1)\cdots\rho_i(t_{n-1})\right)dt_1\cdots dt_{n-1}.$$

The following theorem is a multilinear variant of a splitting theorem, see [6, Lemma 2.23], [19, Lemma 9.14]; such a result is used in the proof of Pietsch's composition theorem for linear operators in [6, Theorem 2.22], [19, Theorem 9.13].

Theorem 3. *Let n be a natural number, $X_1,\ldots,X_n$, Y Banach spaces and $U : X_1 \times \cdots \times X_n \to Y$ a multilinear continuous operator. Let $1 \leqslant k \leqslant n$ and $r_k,\ldots,r_n \in [1,\infty)$, $p_k, q_k,\ldots,p_n, q_n \in (1,\infty)$ be such that*

$$\frac{1}{r_k} = \frac{1}{p_k} + \frac{1}{q_k}, \quad \ldots, \quad \frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{q_n}$$

and let also $q \in (1,\infty)$ be defined by

$$\frac{1}{q^*} = \frac{1}{q_k^*} + \cdots + \frac{1}{q_n^*}.$$

If $p_1,\ldots,p_{k-1} \in [1,\infty)$ (in case $2 \leqslant k \leqslant n$) and U is $(p_1,\ldots,p_n)$-dominated, then for each $(x_i^j)_{1\leqslant i\leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ there exist $(\lambda_i^k)_{1\leqslant i\leqslant m},\ldots,(\lambda_i^n)_{1\leqslant i\leqslant m} \subset \mathbb{K}$, $(y_i)_{1\leqslant i\leqslant m} \subset Y$ such that

$$\begin{aligned}
&l_{p_k}\left(\lambda_i^k \mid 1 \leqslant i \leqslant m\right) \leqslant 1,\ldots,l_{p_n}\left(\lambda_i^n \mid 1 \leqslant i \leqslant m\right) \leqslant 1,\\
&w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_{p_1,\ldots,p_n}(U)B_{p_1}\cdots B_{p_{k-1}}w_2\left(x_i^1 \mid 1 \leqslant i \leqslant m\right)\cdots\\
&\qquad w_2\left(x_i^{k-1} \mid 1 \leqslant i \leqslant m\right)w_{r_k}\left(x_i^k \mid 1 \leqslant i \leqslant m\right)\cdots w_{r_n}\left(x_i^n \mid 1 \leqslant i \leqslant m\right),\\
&U\left(x_i^1,\ldots,x_i^n\right) = \lambda_i^k\cdots\lambda_i^n y_i \quad \text{for each } 1 \leqslant i \leqslant m.
\end{aligned}$$

Above by B_p we denote Khinchin's constant.

Proof. For the case $n = 1$, see [6,19]. Let $n \geqslant 2$. Using a simple argument of homogeneity it is enough to prove that for each $(x_i^j)_{1\leqslant i\leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ with

$$w_2\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \leqslant 1,\ldots,w_2\left(x_i^{k-1} \mid 1 \leqslant i \leqslant m\right) \leqslant 1, \tag{1}$$

$$w_{r_k}\left(x_i^k \mid 1 \leqslant i \leqslant m\right) \leqslant 1,\ldots,w_{r_n}\left(x_i^n \mid 1 \leqslant i \leqslant m\right) \leqslant 1, \tag{2}$$

there exist $(\lambda_i^k)_{1\leqslant i\leqslant m},\ldots,(\lambda_i^n)_{1\leqslant i\leqslant m} \subset \mathbb{K}$, $(y_i)_{1\leqslant i\leqslant m} \subset Y$ such that

$$l_{p_k}\left(\lambda_i^k \mid 1 \leqslant i \leqslant m\right) \leqslant 1,\ldots,l_{p_n}\left(\lambda_i^n \mid 1 \leqslant i \leqslant m\right) \leqslant 1, \tag{3}$$

$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_{p_1,\ldots,p_n}(U)B_{p_1}\cdots B_{p_{k-1}}, \tag{4}$$

$$U\left(x_i^1,\ldots,x_i^n\right) = \lambda_i^k\cdots\lambda_i^n y_i \quad \text{for each } 1 \leqslant i \leqslant m. \tag{5}$$

Since U is $(p_1,\ldots,p_n)$-dominated, by the domination theorem, there exists regular Borel probability measures μ_j on $\Omega_j = B_{X_j^*}$ $(1 \leqslant j \leqslant n)$ such that for each $(x_1,\ldots,x_n) \in X_1 \times \cdots \times X_n$ we have

$$\left\|U(x_1,\ldots,x_n)\right\| \leqslant \delta_{p_1,\ldots,p_n}(U)\left(\int_{\Omega_1}\left|x_1^*(x_1)\right|^{p_1}d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}}\cdots\left(\int_{\Omega_n}\left|x_n^*(x_n)\right|^{p_n}d\mu_n\left(x_n^*\right)\right)^{\frac{1}{p_n}}. \tag{6}$$

In particular, for each $1 \leqslant i \leqslant m$,

$$\left\|U\left(x_i^1, \ldots, x_i^n\right)\right\| \leqslant \delta_{p_1,\ldots,p_n}(U)\left(\int_{\Omega_1}\left|x_1^*\left(x_i^1\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} \cdots \left(\int_{\Omega_n}\left|x_n^*\left(x_i^n\right)\right|^{p_n} d\mu_n\left(x_n^*\right)\right)^{\frac{1}{p_n}}. \tag{7}$$

For each $1 \leqslant i \leqslant m$, we define

$$\lambda_i^k = \left(\int_{\Omega_k}\left|x_k^*\left(x_i^k\right)\right|^{r_k} d\mu_k\left(x_k^*\right)\right)^{\frac{1}{p_k}}, \ldots, \lambda_i^n = \left(\int_{\Omega_n}\left|x_n^*\left(x_i^n\right)\right|^{r_n} d\mu_n\left(x_n^*\right)\right)^{\frac{1}{p_n}}. \tag{8}$$

Then, using (2), we get that (3) are satisfied. Define also

$$y_i = \begin{cases} \frac{U(x_i^1,\ldots,x_i^n)}{\lambda_i^k \cdots \lambda_i^n}, & \text{if all } \lambda_i^k, \ldots, \lambda_i^n \text{ are non-null}, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Let $1 \leqslant i \leqslant m$. If all $\lambda_i^k, \ldots, \lambda_i^n$ are non-null, then from (9) it follows that (5) holds. Otherwise, i.e. there exists $k \leqslant j \leqslant n$ such that $\lambda_i^j = 0$, then by (8),

$$\int_{\Omega_j}\left|x_j^*\left(x_i^j\right)\right|^{r_j} d\mu_j\left(x_j^*\right) = 0,$$

thus $|x_j^*(x_i^j)| = 0$ for μ_j-almost all x_j^* and so also

$$\int_{\Omega_j}\left|x_j^*\left(x_i^j\right)\right|^{p_j} d\mu_j\left(x_j^*\right) = 0,$$

which by (7) gives $U(x_i^1, \ldots, x_i^n) = 0$ and again, using (9), it follows that (5) holds. For the proof of (4), denote

$$I = \left\{1 \leqslant i \leqslant m \;\middle|\; \text{all } \lambda_i^k, \ldots, \lambda_i^n \text{ are non-null}\right\} \subseteq \{1, \ldots, m\}.$$

Then

$$w_q(y_i \mid 1 \leqslant i \leqslant m) = w_q(y_i \mid i \in I).$$

For the convenience of writing we will consider $I = \{1, \ldots, l\}$ with $1 \leqslant l \leqslant m$, which means that for each $1 \leqslant i \leqslant l$ all $\lambda_i^k, \ldots, \lambda_i^n$ are non-null. We use the well-known relation, see [17, Lemma 1.1.14]

$$w_q(y_i \mid 1 \leqslant i \leqslant l) = \sup_{\|(\alpha_1,\ldots,\alpha_l)\|_{q^*} \leqslant 1}\left\|\sum_{i=1}^{l} \alpha_i y_i\right\|$$

and thus

$$w_q(y_i \mid 1 \leqslant i \leqslant m) = \sup_{\|(\alpha_1,\ldots,\alpha_l)\|_{q^*} \leqslant 1}\left\|\sum_{i=1}^{l} \alpha_i y_i\right\|. \tag{10}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be with $\|(\alpha_1, \ldots, \alpha_l)\|_{q^*} \leqslant 1$. In case when $l < m$ we denote $\bar{\alpha} = (\alpha_1, \ldots, \alpha_l, 0, \ldots, 0) \in l_{q^*}^m$ and observe that $\|\bar{\alpha}\|_{q^*} = \|\alpha\|_{q^*} \leqslant 1$. From $\frac{1}{q^*} = \frac{1}{q_k^*} + \cdots + \frac{1}{q_n^*}$, there exist

$$\begin{aligned} &\beta_k = \left(\beta_1^k, \ldots, \beta_m^k\right) \quad \text{with } \|\beta_k\|_{q_k^*} \leqslant 1, \\ &\quad\vdots \\ &\beta_n = \left(\beta_1^n, \ldots, \beta_m^n\right) \quad \text{with } \|\beta_n\|_{q_n^*} \leqslant 1 \end{aligned}$$

such that $\bar{\alpha} = \beta_k \cdots \beta_n$ i.e.

$$\begin{aligned} &\alpha_i = \beta_i^k \cdots \beta_i^n \quad \text{for each } 1 \leqslant i \leqslant l, \\ &\beta_i^k \cdots \beta_i^n = 0 \quad \text{for each } l+1 \leqslant i \leqslant m, \text{ in case } l < m. \end{aligned}$$

Then, by (9) and the fact that U is multilinear we have

$$\sum_{i=1}^{l}\alpha_i y_i=\sum_{i=1}^{l}\beta_i^k\cdots\beta_i^n\frac{U(x_i^1,\ldots,x_i^{k-1},x_i^k,\ldots,x_i^n)}{\lambda_i^k\cdots\lambda_i^n}=\sum_{i=1}^{l}U\left(x_i^1,\ldots,x_i^{k-1},\frac{\beta_i^k x_i^k}{\lambda_i^k},\ldots,\frac{\beta_i^n x_i^n}{\lambda_i^n}\right)$$

which, based on Lemma 2 becomes

$$\sum_{i=1}^{l}\alpha_i y_i=\int\limits_{[0,1]^{n-1}}\cdots\int U\big(x_1(t_1),\ldots,x_{n-1}(t_{n-1}),x_n(t_1,\ldots,t_{n-1})\big)\,dt_1\cdots dt_{n-1}$$

where $x_1:[0,1]\to X_1,\ldots,x_{n-1}:[0,1]\to X_{n-1}$, $x_n:[0,1]^{n-1}\to X_n$ are defined by

$$x_1(t_1)=\sum_{i=1}^{l}x_i^1\rho_i(t_1),\ldots,x_{k-1}(t_{k-1})=\sum_{i=1}^{l}x_i^{k-1}\rho_i(t_{k-1}),$$
$$x_k(t_k)=\sum_{i=1}^{l}\frac{\beta_i^k x_i^k}{\lambda_i^k}\rho_i(t_k),\ldots,x_{n-1}(t_{n-1})=\sum_{i=1}^{l}\frac{\beta_i^{n-1}x_i^{n-1}}{\lambda_i^{n-1}}\rho_i(t_{n-1}),$$
$$x_n(t_1,\ldots,t_{n-1})=\sum_{i=1}^{l}\frac{\beta_i^n x_i^n}{\lambda_i^n}\rho_i(t_1)\cdots\rho_i(t_{n-1}).$$

Then

$$\begin{aligned}\left\|\sum_{i=1}^{l}\alpha_i y_i\right\|&=\left\|\int\limits_{[0,1]^{n-1}}\cdots\int U\big(x_1(t_1),\ldots,x_{n-1}(t_{n-1}),x_n(t_1,\ldots,t_{n-1})\big)\,dt_1\cdots dt_{n-1}\right\|\\&\leqslant\int\limits_{[0,1]^{n-1}}\cdots\int\big\|U\big(x_1(t_1),\ldots,x_{n-1}(t_{n-1}),x_n(t_1,\ldots,t_{n-1})\big)\big\|\,dt_1\cdots dt_{n-1}.\end{aligned}\tag{11}$$

By applying (6) for each $(t_1,\ldots,t_{n-1})\in[0,1]^{n-1}$ we have

$$\begin{aligned}&\big\|U\big(x_1(t_1),\ldots,x_{n-1}(t_{n-1}),x_n(t_1,\ldots,t_{n-1})\big)\big\|\\&\leqslant\delta_{p_1,\ldots,p_n}(U)\left(\int\limits_{\Omega_1}\big|x_1^*\big(x_1(t_1)\big)\big|^{p_1}\,d\mu_1(x_1^*)\right)^{\frac{1}{p_1}}\cdots\left(\int\limits_{\Omega_{n-1}}\big|x_{n-1}^*\big(x_{n-1}(t_{n-1})\big)\big|^{p_{n-1}}\,d\mu_{n-1}\big(x_{n-1}^*\big)\right)^{\frac{1}{p_{n-1}}}\\&\quad\times\left(\int\limits_{\Omega_n}\big|x_n^*\big(x_n(t_1,\ldots,t_{n-1})\big)\big|^{p_n}\,d\mu_n\big(x_n^*\big)\right)^{\frac{1}{p_n}}.\end{aligned}\tag{12}$$

From (11) and (12) we deduce

$$\begin{aligned}\left\|\sum_{i=1}^{l}\alpha_i y_i\right\|&\leqslant\delta_{p_1,\ldots,p_n}(U)\int\limits_{[0,1]^{n-1}}\cdots\int\left(\int\limits_{\Omega_1}\big|x_1^*\big(x_1(t_1)\big)\big|^{p_1}\,d\mu_1\big(x_1^*\big)\right)^{\frac{1}{p_1}}\cdots\\&\quad\times\left(\int\limits_{\Omega_{n-1}}\big|x_{n-1}^*\big(x_{n-1}(t_{n-1})\big)\big|^{p_{n-1}}\,d\mu_{n-1}\big(x_{n-1}^*\big)\right)^{\frac{1}{p_{n-1}}}\\&\quad\times\left(\int\limits_{\Omega_n}\big|x_n^*\big(x_n(t_1,\ldots,t_{n-1})\big)\big|^{p_n}\,d\mu_n\big(x_n^*\big)\right)^{\frac{1}{p_n}}dt_1\cdots dt_{n-1}.\end{aligned}\tag{13}$$

We evaluate the right member in (13) in two stages.

In the first stage we evaluate terms from $k,k+1,\ldots,n$. In order to do so, we use some similar ideas to those used by A. Pietsch in [14, Proof of Theorem 4], [5, Proof of Theorem 11.5], [17, Proof of Theorem 1.3.10].

From $\frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{q_n}$ we deduce $\frac{1}{q_n^*} = \frac{1}{p_n} + \frac{1}{r_n^*}$. Then for each $(t_1, \dots, t_{n-1}) \in [0,1]^{n-1}$ and each $x_n^* \in \Omega_n$ we have

$$\left|x_n^*\left(x_n(t_1, \dots, t_{n-1})\right)\right| \leqslant \sum_{i=1}^{l} \frac{|\beta_i^n||x_n^*(x_i^n)|}{|\lambda_i^n|} = \sum_{i=1}^{l} \frac{|\beta_i^n|^{\frac{q_n^*}{r_n^*}} |\beta_i^n|^{\frac{q_n^*}{p_n}} |x_n^*(x_i^n)|^{\frac{r_n}{p_n}} |x_n^*(x_i^n)|^{\frac{r_n}{q_n}}}{|\lambda_i^n|}.$$

Using that $\frac{1}{r_n^*} + \frac{1}{p_n} + \frac{1}{q_n} = 1$, from Holder's inequality we obtain

$$\sum_{i=1}^{l} \frac{|\beta_i^n|^{\frac{q_n^*}{r_n^*}} |\beta_i^n|^{\frac{q_n^*}{p_n}} |x_n^*(x_i^n)|^{\frac{r_n}{p_n}} |x_n^*(x_i^n)|^{\frac{r_n}{q_n}}}{|\lambda_i^n|} \leqslant \left(\sum_{i=1}^{l} \left|\beta_i^n\right|^{q_n^*}\right)^{\frac{1}{r_n^*}} \left(\sum_{i=1}^{l} \frac{|\beta_i^n|^{q_n^*} |x_n^*(x_i^n)|^{r_n}}{|\lambda_i^n|^{p_n}}\right)^{\frac{1}{p_n}} \left(\sum_{i=1}^{l} \left|x_n^*\left(x_i^n\right)\right|^{r_n}\right)^{\frac{1}{q_n}}$$

which if we observe that by (2)

$$\sum_{i=1}^{l} \left|x_n^*\left(x_i^n\right)\right|^{r_n} \leqslant \sum_{i=1}^{m} \left|x_n^*\left(x_i^n\right)\right|^{r_n} = \left[w_{r_n}\left(x_i^n \mid 1 \leqslant i \leqslant m\right)\right]^{r_n} \leqslant 1$$

and

$$\sum_{i=1}^{l} \left|\beta_i^n\right|^{q_n^*} \leqslant \sum_{i=1}^{m} \left|\beta_i^n\right|^{q_n^*} = \|\beta_n\|_{q_n^*}^{q_n^*} \leqslant 1$$

gives

$$\left|x_n^*\left(x_n(t_1, \dots, t_{n-1})\right)\right| \leqslant \left(\sum_{i=1}^{l} \frac{|\beta_i^n|^{q_n^*} |x_n^*(x_i^n)|^{r_n}}{|\lambda_i^n|^{p_n}}\right)^{\frac{1}{p_n}}.$$

From here, by integration and definition of λ_i^n given in (8), we obtain

$$\int_{\Omega_n} \left|x_n^*\left(x_n(t_1, \dots, t_{n-1})\right)\right|^{p_n} d\mu_n\left(x_n^*\right) \leqslant \int_{\Omega_n} \sum_{i=1}^{l} \frac{|\beta_i^n|^{q_n^*} |x_n^*(x_i^n)|^{r_n}}{|\lambda_i^n|^{p_n}} d\mu_n\left(x_n^*\right) = \sum_{i=1}^{l} \left|\beta_i^n\right|^{q_n^*} \cdot \frac{\int_{\Omega_n} |x_n^*(x_i^n)|^{r_n} d\mu_n(x_n^*)}{|\lambda_i^n|^{p_n}}$$
$$= \sum_{i=1}^{l} \left|\beta_i^n\right|^{q_n^*} \leqslant \sum_{i=1}^{m} \left|\beta_i^n\right|^{q_n^*} = \|\beta_n\|_{q_n^*}^{q_n^*} \leqslant 1.$$

Thus

$$\int_{\Omega_n} \left|x_n^*\left(x_n(t_1, \dots, t_{n-1})\right)\right|^{p_n} d\mu_n\left(x_n^*\right) \leqslant 1 \quad \text{for each } (t_1, \dots, t_{n-1}) \in [0,1]^{n-1}.$$

In the same way, by using $\frac{1}{r_k} = \frac{1}{p_k} + \frac{1}{q_k}, \dots, \frac{1}{r_{n-1}} = \frac{1}{p_{n-1}} + \frac{1}{q_{n-1}}$ we can prove that

$$\left(\int_{\Omega_k} \left|x_k^*\left(x_k(t_k)\right)\right|^{p_k} d\mu_k\left(x_k^*\right)\right)^{\frac{1}{p_k}} \leqslant 1 \quad \text{for each } t_k \in [0,1], \dots,$$

$$\left(\int_{\Omega_{n-1}} \left|x_{n-1}^*\left(x_{n-1}(t_{n-1})\right)\right|^{p_{n-1}} d\mu_{n-1}\left(x_{n-1}^*\right)\right)^{\frac{1}{p_{n-1}}} \leqslant 1 \quad \text{for each } t_{n-1} \in [0,1].$$

Using these inequalities in (13) we get

$$\left\|\sum_{i=1}^{l} \alpha_i y_i\right\| \leqslant \delta_{p_1, \dots, p_n}(U) \int_{[0,1]^{k-1}} \cdots \int \left(\int_{\Omega_1} \left|x_1^*\left(x_1(t_1)\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} \cdots$$
$$\left(\int_{\Omega_{k-1}} \left|x_{k-1}^*\left(x_{k-1}(t_{k-1})\right)\right|^{p_{k-1}} d\mu_{k-1}\left(x_{k-1}^*\right)\right)^{\frac{1}{p_{k-1}}} dt_1 \cdots dt_{k-1}. \tag{14}$$

Here we must remark that in the case when $k = 1$ these terms do not occur and the proof will be finished.

For $k \geqslant 2$ we evaluate the right member of (14), in a second stage.
Using Fubini's theorem in the right member in (14) we obtain

$$\left\| \sum_{i=1}^{l} \alpha_i y_i \right\| \leqslant \delta_{p_1,\dots,p_n}(U) \left[\int_0^1 \left(\int_{\Omega_1} \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} dt_1 \right] \cdots \left[\int_0^1 \left(\int_{\Omega_{k-1}} \left| x_{k-1}^*\left(x_{k-1}(t_{k-1})\right)\right|^{p_{k-1}} d\mu_{k-1}\left(x_{k-1}^*\right)\right)^{\frac{1}{p_{k-1}}} dt_{k-1} \right]. \tag{15}$$

From $\| \ \|_{L_1[0,1]} \leqslant \| \ \|_{L_{p_1}[0,1]}$ and Fubini's theorem we have

$$\int_0^1 \left(\int_{\Omega_1} \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} dt_1 \leqslant \left(\int_0^1 \left(\int_{\Omega_1} \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right) dt_1 \right)^{\frac{1}{p_1}}$$
$$= \left(\int_{\Omega_1} \left(\int_0^1 \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} dt_1 \right) d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}}.$$

For each $x_1^* \in \Omega_1$, by Khinchin's inequality we have

$$\int_0^1 \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} dt_1 = \int_0^1 \left| \sum_{i=1}^{l} x_1^*\left(x_i^1\right) \rho_i(t_1)\right|^{p_1} dt_1 \leqslant [B_{p_1}]^{p_1} \left(\sum_{i=1}^{l} \left| x_1^*\left(x_i^1\right)\right|^2 \right)^{\frac{p_1}{2}}.$$

On the other hand, by using (1), we obtain

$$\sum_{i=1}^{l} \left| x_1^*\left(x_i^1\right)\right|^2 \leqslant \sum_{i=1}^{m} \left| x_1^*\left(x_i^1\right)\right|^2 \leqslant \left[w_2\left(x_i^1 \mid 1 \leqslant i \leqslant m\right)\right]^2 \leqslant 1$$

and then

$$\int_0^1 \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} dt_1 \leqslant [B_{p_1}]^{p_1}.$$

Since μ_1 is a probability, it follows

$$\left(\int_{\Omega_1} \left(\int_0^1 \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} dt_1 \right) d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} \leqslant B_{p_1}$$

thus

$$\int_0^1 \left(\int_{\Omega_1} \left| x_1^*\left(x_1(t_1)\right)\right|^{p_1} d\mu_1\left(x_1^*\right)\right)^{\frac{1}{p_1}} dt_1 \leqslant B_{p_1}.$$

In the same way we can prove

$$\int_0^1 \left(\int_{\Omega_2} \left| x_2^*\left(x_2(t_2)\right)\right|^{p_2} d\mu_2\left(x_2^*\right)\right)^{\frac{1}{p_2}} dt_2 \leqslant B_{p_2},$$
$$\vdots$$
$$\int_0^1 \left(\int_{\Omega_{k-1}} \left| x_{k-1}^*\left(x_{k-1}(t_{k-1})\right)\right|^{p_{k-1}} d\mu_{k-1}\left(x_{k-1}^*\right)\right)^{\frac{1}{p_{k-1}}} dt_{k-1} \leqslant B_{p_{k-1}}.$$

Using these inequalities in (15) we deduce that

$$\left\| \sum_{i=1}^{l} \alpha_i y_i \right\| \leqslant \delta_{p_1,\ldots,p_n}(U) B_{p_1} \cdots B_{p_{k-1}}$$

which, by relation (10), proves that (4) also holds. $\square$

Theorem 3 has many consequences which will be studied next. First of all, we prove that in case $r = 1$ the answer to Question is **YES** for all natural numbers.

Theorem 4. *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all natural numbers n we have*

$$\pi_q \circ \delta_p^n \subset \pi_1^n.$$

Proof. Let $X_1 \times \cdots \times X_n \xrightarrow{U} Y \xrightarrow{T} Z$ be a diagram, where U is p-dominated and T is q-summing. Because $p \leqslant np$ and U is p-dominated, from the inclusion theorem, U is np-dominated and

$$\delta_{np}(U) \leqslant \delta_p(U). \tag{1}$$

Next we consider in Theorem 3, $k = 1$, $p_1 = \cdots = p_n = np$, $q_1 = \cdots = q_n = (np)^*$, $r_1 = \cdots = r_n = 1$; we also have $\frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*} = \frac{1}{p} = \frac{1}{q^*}$.

Let $1 \leqslant j \leqslant n$ and $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$. By Theorem 3 there exist $(\lambda_i^1)_{1 \leqslant i \leqslant m}, \ldots, (\lambda_i^n)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that

$$l_{np}\left(\lambda_i^1 \mid 1 \leqslant i \leqslant m\right) \leqslant 1, \ldots, l_{np}\left(\lambda_i^n \mid 1 \leqslant i \leqslant m\right) \leqslant 1, \tag{2}$$

$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_{np}(U) w_1\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_1\left(x_i^n \mid 1 \leqslant i \leqslant m\right), \tag{3}$$

$$U\left(x_i^1, \ldots, x_i^n\right) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \text{for each } 1 \leqslant i \leqslant m. \tag{4}$$

From (4), $\underbrace{\frac{1}{np} + \cdots + \frac{1}{np}}_{n \text{ times}} + \frac{1}{q} = 1$, Holder's inequality and (2) we get

$$\begin{aligned} l_1\left(T\left(U\left(x_i^1, \ldots, x_i^n\right)\right) \mid 1 \leqslant i \leqslant m\right) &= l_1\left(\lambda_i^1 \cdots \lambda_i^n T(y_i) \mid 1 \leqslant i \leqslant m\right) \\ &\leqslant l_{np}\left(\lambda_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots l_{np}\left(\lambda_i^n \mid 1 \leqslant i \leqslant m\right) l_q\left(T(y_i) \mid 1 \leqslant i \leqslant m\right) \\ &\leqslant l_q\left(T(y_i) \mid 1 \leqslant i \leqslant m\right). \end{aligned} \tag{5}$$

Since T is q-summing

$$l_q\left(T(y_i) \mid 1 \leqslant i \leqslant m\right) \leqslant \pi_q(T) w_q(y_i \mid 1 \leqslant i \leqslant m). \tag{6}$$

By using (6) and (3), in (5), we deduce

$$l_1\left(T\left(U\left(x_i^1, \ldots, x_i^n\right)\right) \mid 1 \leqslant i \leqslant m\right) \leqslant \pi_q(T) \delta_{np}(U) w_1\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_1\left(x_i^n \mid 1 \leqslant i \leqslant m\right),$$

which by (1) gives

$$l_1\left(T\left(U\left(x_i^1, \ldots, x_i^n\right)\right) \mid 1 \leqslant i \leqslant m\right) \leqslant \pi_q(T) \delta_p(U) w_1\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_1\left(x_i^n \mid 1 \leqslant i \leqslant m\right)$$

i.e. $T \circ U$ is 1-summing and

$$\pi_1(T \circ U) \leqslant \delta_p(U) \pi_q(T). \qquad \square$$

We state two consequences of Theorem 3. The first one is the corresponding one to the case $k = 1$. As it is expected this situation has important applications.

Corollary 5. *Let n be a natural number, $r_1, \ldots, r_n \in [1, \infty)$, $p_1, q_1, \ldots, p_n, q_n \in (1, \infty)$ such that*

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \quad \ldots, \quad \frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{q_n}$$

and let also $q \in (1, \infty)$ be defined by $\frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*}$.

If $U : X_1 \times \cdots \times X_n \to Y$ is $(p_1, \ldots, p_n)$-dominated, then for each $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ there exists $(\lambda_i^j)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$ $(1 \leqslant j \leqslant n)$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that

$$l_{p_1}\big(\lambda_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1, \ldots, l_{p_n}\big(\lambda_i^n \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1,$$
$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_{p_1,\ldots,p_n}(U) w_{r_1}\big(x_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \cdots w_{r_n}\big(x_i^n \,\big|\, 1 \leqslant i \leqslant m\big),$$
$$U\big(x_i^1, \ldots, x_i^n\big) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \textit{for each } 1 \leqslant i \leqslant m.$$

In order to give some consequences of Corollary 5 we make:

Observation 6. Suppose $p_1 = \cdots = p_n = p \in (1, \infty)$ and the conditions

$$\frac{1}{r_1} = \frac{1}{p} + \frac{1}{q_1}, \quad \ldots, \quad \frac{1}{r_n} = \frac{1}{p} + \frac{1}{q_n}, \quad \frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*},$$

are satisfied. Then

(i) $\frac{1}{q_1^*} = \frac{1}{p} + \frac{1}{r_1^*}, \ldots, \frac{1}{q_n^*} = \frac{1}{p} + \frac{1}{r_n^*}$, which by addition gives $\frac{1}{r_1^*} + \cdots + \frac{1}{r_n^*} = \frac{1}{q^*} - \frac{n}{p}$ and this forces $p \geqslant nq^*$.

(ii) The equality $q_1 = \cdots = q_n = \beta \in (1, \infty)$ is equivalent to $r_1 = \cdots = r_n = \alpha \in [1, \infty)$ and $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{\beta}$. Further the condition $\frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*}$ is equivalent to $\frac{1}{q^*} = \frac{n}{\beta^*}$, $\beta = (nq^*)^*$ i.e. $\beta = q_1 = \cdots = q_n = (nq^*)^*$ and by (i), $p \geqslant nq^*$.

Based on Observation 6(i) and (ii), Corollary 5 implies:

Corollary 7. *Let n be a natural number, $p, q \in (1, \infty)$ such that $p \geqslant nq^*$.*

(i) *Let $r_1, \ldots, r_n \in [1, \infty)$, $q_1, \ldots, q_n \in (1, \infty)$ be such that*

$$\frac{1}{r_1} - \frac{1}{q_1} = \cdots = \frac{1}{r_n} - \frac{1}{q_n} = \frac{1}{p}, \qquad \frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*}.$$

If $U : X_1 \times \cdots \times X_n \to Y$ is p-dominated, then for each $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ there exists $(\lambda_i^j)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$ $(1 \leqslant j \leqslant n)$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that

$$l_p\big(\lambda_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1, \ldots, l_p\big(\lambda_i^n \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1,$$
$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_p(U) w_{r_1}\big(x_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \cdots w_{r_n}\big(x_i^n \,\big|\, 1 \leqslant i \leqslant m\big),$$
$$U\big(x_i^1, \ldots, x_i^n\big) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \textit{for each } 1 \leqslant i \leqslant m.$$

(ii) *Let n be a natural number, $p, q \in (1, \infty)$ such that $p \geqslant nq^*$ and define α by $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{(nq^*)^*}$. If $U : X_1 \times \cdots \times X_n \to Y$ is p-dominated, then for each $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ there exists $(\lambda_i^j)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$ $(1 \leqslant j \leqslant n)$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that*

$$l_p\big(\lambda_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1, \ldots, l_p\big(\lambda_i^n \,\big|\, 1 \leqslant i \leqslant m\big) \leqslant 1,$$
$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_p(U) w_\alpha\big(x_i^1 \,\big|\, 1 \leqslant i \leqslant m\big) \cdots w_\alpha\big(x_i^n \,\big|\, 1 \leqslant i \leqslant m\big),$$
$$U\big(x_i^1, \ldots, x_i^n\big) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \textit{for each } 1 \leqslant i \leqslant m.$$

Following a suggestion of the referee we give a concrete situation of above corollary. By taking $p = 4$ in Corollary 7(i), we have $n \leqslant \frac{4}{q^*} < 4$, hence excluding the linear case we must have $n = 2$ or $n = 3$. We state just the case $n = 2$, when $q \geqslant 2$.

Corollary 8. *Let $r_1, r_2 \in [1, \infty)$, $q_1, q_2 \in (1, \infty)$, $q \in [2, \infty)$ be such that*

$$\frac{1}{r_1} = \frac{1}{q_1} + \frac{1}{4}, \qquad \frac{1}{r_2} = \frac{1}{q_2} + \frac{1}{4}, \qquad \frac{1}{q^*} = \frac{1}{q_1^*} + \frac{1}{q_2^*}.$$

If $U : X \times Y \to Z$ is 4-dominated, then for each $(x_i)_{1 \leqslant i \leqslant m} \subset X$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ there exists $(\lambda_i)_{1 \leqslant i \leqslant m}$, $(\nu_i)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$, $(z_i)_{1 \leqslant i \leqslant m} \subset Z$ such that

$$l_4(\lambda_i \mid 1 \leqslant i \leqslant m) \leqslant 1, \qquad l_4(\nu_i \mid 1 \leqslant i \leqslant m) \leqslant 1,$$
$$w_q(z_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_4(U) w_{r_1}(x_i \mid 1 \leqslant i \leqslant m) w_{r_2}(y_i \mid 1 \leqslant i \leqslant m),$$
$$U(x_i, y_i) = \lambda_i \nu_i z_i \quad \textit{for each } 1 \leqslant i \leqslant m.$$

3. Pietsch composition results

In the sequel we derive from Theorem 3, as in the linear case, some results of Pietsch's composition type. First we analyze some consequences of Corollary 5.

Theorem 9. *Let n be a natural number, $r_1, \dots, r_n \in [1, \infty)$, $p_1, q_1, \dots, p_n, q_n \in (1, \infty)$ such that*

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \quad \dots, \quad \frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{q_n}$$

and let also $q \in (1, \infty)$ be defined by $\frac{1}{q^*} = \frac{1}{q_1^*} + \dots + \frac{1}{q_n^*}$. Then

$$\pi_q \circ \delta_{p_1,\dots,p_n} \subset \pi_{t;r_1,\dots,r_n}, \quad \text{where } \frac{1}{t} = \frac{1}{p_1} + \dots + \frac{1}{p_n} + \frac{1}{q}.$$

Proof. We prove that the condition $\frac{1}{t} \leqslant \frac{1}{r_1} + \dots + \frac{1}{r_n}$ is verified. Indeed, by definition of t, this is equivalent to $\frac{1}{q} \leqslant (\frac{1}{r_1} - \frac{1}{p_1}) + \dots + (\frac{1}{r_n} - \frac{1}{p_n})$, which by hypothesis is equivalent to $\frac{1}{q} \leqslant \frac{1}{q_1} + \dots + \frac{1}{q_n}$. This holds because, $\frac{1}{q^*} = \frac{1}{q_1^*} + \dots + \frac{1}{q_n^*}$ is equivalent to $\frac{1}{q} - (\frac{1}{q_1} + \dots + \frac{1}{q_n}) = 1 - n \leqslant 0$. Let $X_1 \times \dots \times X_n \overset{U}{\to} Y \overset{T}{\to} Z$ be a diagram, where U is $(p_1, \dots, p_n)$-dominated and T is q-summing. Let $1 \leqslant j \leqslant n$ and $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$. By Corollary 5 there exist $(\lambda_i^j)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that

$$l_{p_1}(\lambda_i^1 \mid 1 \leqslant i \leqslant m) \leqslant 1, \dots, l_{p_n}(\lambda_i^n \mid 1 \leqslant i \leqslant m) \leqslant 1, \tag{1}$$

$$w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_{p_1,\dots,p_n}(U) w_{r_1}(x_i^1 \mid 1 \leqslant i \leqslant m) \cdots w_{r_n}(x_i^n \mid 1 \leqslant i \leqslant m), \tag{2}$$

$$U(x_i^1, \dots, x_i^n) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \text{for each } 1 \leqslant i \leqslant m. \tag{3}$$

From (3), $\frac{1}{t} = \frac{1}{p_1} + \dots + \frac{1}{p_n} + \frac{1}{q}$, Holder's inequality and (1) we have

$$\begin{aligned} l_t\big(T\big(U\big(x_i^1, \dots, x_i^n\big)\big) \mid 1 \leqslant i \leqslant m\big) &= l_t\big(\lambda_i^1 \cdots \lambda_i^n T(y_i) \mid 1 \leqslant i \leqslant m\big) \\ &\leqslant l_{p_1}\big(\lambda_i^1 \mid 1 \leqslant i \leqslant m\big) \cdots l_{p_n}\big(\lambda_i^n \mid 1 \leqslant i \leqslant m\big) l_q\big(T(y_i) \mid 1 \leqslant i \leqslant m\big) \\ &\leqslant l_q\big(T(y_i) \mid 1 \leqslant i \leqslant m\big). \end{aligned} \tag{4}$$

Since T is q-summing

$$l_q\big(T(y_i) \mid 1 \leqslant i \leqslant m\big) \leqslant \pi_q(T) w_q(y_i \mid 1 \leqslant i \leqslant m). \tag{5}$$

Based on (5) and (2), in (4), we deduce

$$l_t\big(T\big(U\big(x_i^1, \dots, x_i^n\big)\big) \mid 1 \leqslant i \leqslant m\big) \leqslant \pi_q(T)\delta_{p_1,\dots,p_n}(U) w_{r_1}\big(x_i^1 \mid 1 \leqslant i \leqslant m\big) \cdots w_{r_n}\big(x_i^n \mid 1 \leqslant i \leqslant m\big),$$

which by definition means that $T \circ U$ is $(t; r_1, \dots, r_n)$-summing and

$$\pi_{t;r_1,\dots,r_n}(T \circ U) \leqslant \delta_{p_1,\dots,p_n}(U)\pi_q(T). \qquad \square$$

In case $p_1 = \dots = p_n = p \in (1, \infty)$ from Theorem 9 and Observation 6(i) and (ii) we obtain:

Corollary 10. *Let n be a natural number, $p, q \in (1, \infty)$ such that $p \geqslant nq^*$.*

(i) *Let $r_1, \dots, r_n \in [1, \infty)$, $q_1, \dots, q_n \in (1, \infty)$ be such that*

$$\frac{1}{r_1} - \frac{1}{q_1} = \dots = \frac{1}{r_n} - \frac{1}{q_n} = \frac{1}{p}, \qquad \frac{1}{q^*} = \frac{1}{q_1^*} + \dots + \frac{1}{q_n^*}.$$

Then

$$\pi_q \circ \delta_p \subset \pi_{t;r_1,\dots,r_n}, \quad \text{where } \frac{1}{t} = \frac{n}{p} + \frac{1}{q}.$$

(ii) *Define α by $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{(nq^*)^*}$. Then*

$$\pi_q \circ \delta_p \subset \pi_{t;\alpha}, \quad \text{where } \frac{1}{t} = \frac{n}{p} + \frac{1}{q}.$$

Remark 11. A natural question is when do we have $t = \alpha$ in Corollary 10(ii)? This is true if and only if $\frac{n}{p} + \frac{1}{q} = \frac{1}{p} + \frac{1}{(nq^*)^*}$ i.e. $\frac{n-1}{p} = 1 - \frac{1}{q} - \frac{1}{nq^*}$, $\frac{n-1}{p} = \frac{n-1}{nq^*}$ which is true when $n = 1$ i.e. we are in the linear case, or when $p = nq^*$, all natural numbers n, in which case $\alpha = t = 1$. Thus from Corollary 10(ii) we get:

Corollary 12. *Let n be a natural number and $q \in (1, \infty)$. Then*

$$\pi_q \circ \delta_{nq^*} \subset \pi_1.$$

Corollary 13.

(i) *Let n be a natural number, $r \in [1, \infty)$, $q \in (1, \infty)$, $p_1, \ldots, p_n \in (1, \infty)$ such that $r < p_1, \ldots, r < p_n$ and*

$$\frac{1}{q^*} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} + \frac{n}{r^*}.$$

Then

$$\pi_q \circ \delta_{p_1,\ldots,p_n} \subset \pi_{(\frac{r^*}{n})^*;r}.$$

(ii) *Let n be a natural number, $r \in [1, \infty)$, $p, q \in (1, \infty)$ such that $p \geqslant nq^*$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{nq} + \frac{1}{n^*}$, or equivalently, $\frac{1}{q^*} = \frac{n}{p} + \frac{n}{r^*}$. Then*

$$\pi_q \circ \delta_p \subset \pi_{(\frac{r^*}{n})^*;r}.$$

Above for $r = 1$ we consider $(\frac{r^*}{n})^* = 1$.

Proof. (i) By taking in Theorem 9, $r_1 = \cdots = r_n = r$, $\frac{1}{q_1} = \frac{1}{r} - \frac{1}{p_1}, \ldots, \frac{1}{q_n} = \frac{1}{r} - \frac{1}{p_n}$, we note that under our hypotheses the condition $\frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*}$ is verified. Then, $\pi_q \circ \delta_{p_1,\ldots,p_n} \subset \pi_{t;r}$, where $\frac{1}{t} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} + \frac{1}{q}$. From $\frac{1}{t} = \frac{1}{q^*} + \frac{1}{q} - \frac{n}{r^*} = 1 - \frac{n}{r^*}$, the statement follows.

(ii) Take $p_1 = \cdots = p_n = p$ and $q_1 = \cdots = q_n = \frac{1}{\frac{1}{r} - \frac{1}{p}}$. Then $\frac{1}{q_1^*} = \cdots = \frac{1}{q_n^*} = \frac{1}{p} + \frac{1}{r^*}$ and $\frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*} = \frac{n}{p} + \frac{n}{r^*} = \frac{1}{q^*}$. By Theorem 9, $\pi_q \circ \delta_p \subset \pi_{t;r}$, where $\frac{1}{t} = \frac{n}{p} + \frac{1}{q}$. From hypothesis, $\frac{1}{t} = \frac{1}{q^*} - \frac{n}{r^*} + \frac{1}{q} = 1 - \frac{n}{r^*}$ and the statement follows. □

Following again a suggestion of the referee we give a concrete situation of above corollary.

Corollary 14.

(i) *Let n be a natural number, $q \in (1, \infty)$, $p_1, \ldots, p_n \in (1, \infty)$ such that $(2n)^* < p_1, \ldots, (2n)^* < p_n$ and*

$$\frac{1}{q} + \frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{2}.$$

Then $\pi_q \circ \delta_{p_1,\ldots,p_n} \subset \pi_{2;(2n)^*}$.

(ii) *For each natural number n we have $\pi_q \circ \delta_{q^2,q^3,\ldots,q^{n+1}} \subset \pi_{2;(2n)^*}$, where $q > 2$ is the only one solution of the equation $\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{n+1}} = \frac{1}{2}$.*

(iii) *For each natural number n we have $\pi_{2^{n+1}} \circ \delta_{2^2,2^3,\ldots,2^{n+1}} \subset \pi_{2;(2n)^*}$.*

Proof. Indeed, (i) follows from Corollary 13(i) for $r = (2n)^*$, while (ii) and (iii) are particular cases of (i). □

We state some possible multilinear variants of Pietsch composition theorem. We include here, as a particular case of Theorem 9, the answer to Question in case $r = 1$.

Corollary 15.

(i) *Let n be a natural number, $q_1, \ldots, q_n \in (1, \infty)$ and $q \in (1, \infty)$ such that $\frac{1}{q^*} = \frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*}$. Then $\pi_q \circ \delta_{q_1^*,\ldots,q_n^*} \subset \pi_1$.*

(ii) *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all natural numbers n we have $\pi_q \circ \delta_p \subset \pi_1$.*

(iii) *Let n be a natural number, $p, q \in (1, \infty)$ and $r \in [1, \infty)$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $p < n^*q^*$, or equivalently $r^* > nq^*$, then $\pi_q \circ \delta_p \subset \pi_{(\frac{r^*}{n})^*,r}$, where for $r = 1$ we consider $(\frac{r^*}{n})^* = 1$.*

(iv) *Let n be a natural number, $r \in [1, \infty)$, $p, q \in (1, \infty)$ such that $\frac{1}{r} = \frac{n}{p} + \frac{1}{q}$. Then $\pi_q \circ \delta_p \subset \pi_{r,(nr^*)^*}$.*

Proof. (i) By taking in Theorem 9, $r_1 = \cdots = r_n = 1$ and $p_1 = q_1^*, \ldots, p_n = q_n^*$, we observe that, by hypothesis, we obtain $t = 1$.

(ii) Let us consider in Theorem 9, $q_1^* = \cdots = q_n^* = np$. Then $\frac{1}{q_1^*} + \cdots + \frac{1}{q_n^*} = \frac{1}{p} = \frac{1}{q^*}$ and by (i) $\pi_q \circ \delta_{np} \subset \pi_1$. From $p \leqslant np$, by inclusion theorem, $\delta_p \subset \delta_{np}$, thus $\pi_q \circ \delta_p \subset \pi_q \circ \delta_{np}$ and so $\pi_q \circ \delta_p \subset \pi_1$. In fact this was the argument which we have used in the direct proof of Theorem 4.

(iii) Under our hypothesis we define s by $\frac{1}{s} = \frac{1}{nq^*} - \frac{1}{r^*}$ and observe that $s \in [1, \infty)$. By taking in Theorem 9, $r_1 = \cdots = r_n = r$, $p_1 = \cdots = p_n = s$, $q_1 = \cdots = q_n = (nq^*)^*$, we obtain $t = (\frac{r^*}{n})^*$.

(iv) By taking in Theorem 9, $r_1 = \cdots = r_n = (nr^*)^*$, $p_1 = \cdots = p_n = p$, $q_1 = \cdots = q_n = (nq^*)^*$ and observe that from $\frac{1}{r} = \frac{n}{p} + \frac{1}{q}$, the hypotheses are satisfied. In this situation we obtain $t = r$. □

Remark 16. In Corollaries 10, 12–15, our main goal was only to state multilinear variants of Pietsch's composition theorem. We must remark that in each of these situations, under the same assumptions, we can obtain a corollary of splitting Theorem 3. For the convenience of the reader we state and prove one of them.

Corollary 17. *Let $r \in [1, \infty)$, $p, q \in (1, \infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and n a natural number with $p < n^*q^*$, or equivalently $r^* > nq^*$ and define $s \in [1, \infty)$ by $\frac{1}{s} = \frac{1}{nq^*} - \frac{1}{r^*}$. If $U : X_1 \times \cdots \times X_n \to Y$ is p-dominated, then for each $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ $(1 \leqslant j \leqslant n)$ there exists $(\lambda_i^j)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$ $(1 \leqslant j \leqslant n)$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that*

$$
\begin{aligned}
&l_s\big(\lambda_i^1 \bigm| 1 \leqslant i \leqslant m\big) \leqslant 1, \ldots, l_s\big(\lambda_i^n \bigm| 1 \leqslant i \leqslant m\big) \leqslant 1, \\
&w_q(y_i \mid 1 \leqslant i \leqslant m) \leqslant \delta_p(U) w_r\big(x_i^1 \bigm| 1 \leqslant i \leqslant m\big) \cdots w_r\big(x_i^n \bigm| 1 \leqslant i \leqslant m\big), \\
&U\big(x_i^1, \ldots, x_i^n\big) = \lambda_i^1 \cdots \lambda_i^n y_i \quad \textit{for each } 1 \leqslant i \leqslant m.
\end{aligned}
$$

Proof. Take as in Corollary 15(iii), s defined by $\frac{1}{s} = \frac{1}{nq^*} - \frac{1}{r^*}$ and observe that $s \in [1, \infty)$. Next, we apply Theorem 3 for $r_1 = \cdots = r_n = r$, $p_1 = \cdots = p_n = s$, $q_1 = \cdots = q_n = (nq^*)^*$. □

In the sequel we prove another Pietsch composition result which is obtained by applying the general form of Theorem 3.

Theorem 18. *Let n, k be two natural number such that $1 \leqslant k \leqslant n$, $r_k, \ldots, r_n \in [1, \infty)$, $p_k, q_k, \ldots, p_n, q_n \in (1, \infty)$ such that*

$$\frac{1}{r_k} = \frac{1}{p_k} + \frac{1}{q_k}, \quad \ldots, \quad \frac{1}{r_n} = \frac{1}{p_n} + \frac{1}{q_n}$$

and let also $q \in (1, \infty)$ be defined by $\frac{1}{q^*} = \frac{1}{q_k^*} + \cdots + \frac{1}{q_n^*}$.

If $p_1, \ldots, p_{k-1} \in [1, \infty)$ (in case $2 \leqslant k \leqslant n$), then

$$\pi_q \circ \delta_{p_1, \ldots, p_n} \subset \pi_{t; \underbrace{2, \ldots, 2}_{k-1 \text{ times}}, r_k, \ldots, r_n}, \quad \textit{where } \frac{1}{t} = \frac{1}{p_k} + \cdots + \frac{1}{p_n} + \frac{1}{q}.$$

Proof. We prove that the condition $\frac{1}{t} \leqslant \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{k-1 \text{ times}} + \frac{1}{r_k} + \cdots + \frac{1}{r_n}$ is satisfied. Indeed, by definition of t and hypothesis, this is equivalent to

$$\frac{1}{q} - \left(\frac{1}{q_k} + \cdots + \frac{1}{q_n}\right) \leqslant \frac{k-1}{2}.$$

But this is true, because $\frac{1}{q} - (\frac{1}{q_k} + \cdots + \frac{1}{q_n}) = 1 - \frac{1}{q^*} - [(1 - \frac{1}{q_k^*}) + \cdots + (1 - \frac{1}{q_n^*})] = k - n$ and further $k - n \leqslant 0 \leqslant \frac{k-1}{2}$. Let $X_1 \times \cdots \times X_n \xrightarrow{U} Y \xrightarrow{T} Z$ be a diagram, where U is $(p_1, \ldots, p_n)$-dominated and T is q-summing.

Let $1 \leqslant j \leqslant n$ and $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$. By Theorem 3, there exist $(\lambda_i^k)_{1 \leqslant i \leqslant m} \subset \mathbb{K}, \ldots, (\lambda_i^n)_{1 \leqslant i \leqslant m} \subset \mathbb{K}$, $(y_i)_{1 \leqslant i \leqslant m} \subset Y$ such that

$$l_{p_k}\big(\lambda_i^k \bigm| 1 \leqslant i \leqslant m\big) \leqslant 1, \ldots, l_{p_n}\big(\lambda_i^n \bigm| 1 \leqslant i \leqslant m\big) \leqslant 1, \tag{1}$$

$$
\begin{aligned}
w_q(y_i \mid 1 \leqslant i \leqslant m) &\leqslant \delta_{p_1, \ldots, p_n}(U) B_{p_1} \cdots B_{p_{k-1}} w_2\big(x_i^1 \bigm| 1 \leqslant i \leqslant m\big) \cdots \\
&\quad w_2\big(x_i^{k-1} \bigm| 1 \leqslant i \leqslant m\big) w_{r_k}\big(x_i^k \bigm| 1 \leqslant i \leqslant m\big) \cdots w_{r_n}\big(x_i^n \bigm| 1 \leqslant i \leqslant m\big),
\end{aligned} \tag{2}
$$

$$U\big(x_i^1, \ldots, x_i^n\big) = \lambda_i^k \cdots \lambda_i^n y_i \quad \text{for each } 1 \leqslant i \leqslant m. \tag{3}$$

From (3), $\frac{1}{t} = \frac{1}{p_k} + \cdots + \frac{1}{p_n} + \frac{1}{q}$, Holder's inequality and (1) we have

$$l_t\big(T\big(U\big(x_i^1, \ldots, x_i^n\big)\big) \bigm| 1 \leqslant i \leqslant m\big) = l_t\big(\lambda_i^k \cdots \lambda_i^n T(y_i) \bigm| 1 \leqslant i \leqslant m\big)$$
$$\leqslant l_{p_k}\big(\lambda_i^k \bigm| 1 \leqslant i \leqslant m\big) \cdots l_{p_n}\big(\lambda_i^n \bigm| 1 \leqslant i \leqslant m\big) l_q\big(T(y_i) \bigm| 1 \leqslant i \leqslant m\big)$$
$$\leqslant l_q\big(T(y_i) \bigm| 1 \leqslant i \leqslant m\big). \tag{4}$$

Since T is q-summing,

$$l_q\big(T(y_i) \bigm| 1 \leqslant i \leqslant m\big) \leqslant \pi_q(T) w_q(y_i \mid 1 \leqslant i \leqslant m). \tag{5}$$

Based on (5) and (4) we deduce

$$l_t\big(T\big(U\big(x_i^1, \ldots, x_i^n\big)\big) \bigm| 1 \leqslant i \leqslant m\big) \leqslant \pi_q(T) w_q(y_i \mid 1 \leqslant i \leqslant m),$$

which if we use (2) gives

$$l_t\big(T\big(U\big(x_i^1, \ldots, x_i^n\big)\big) \bigm| 1 \leqslant i \leqslant m\big)$$
$$\leqslant \pi_q(T)\delta_{p_1,\ldots,p_n}(U) B_{p_1} \cdots B_{p_{k-1}} \cdot w_2\big(x_i^1 \bigm| 1 \leqslant i \leqslant m\big) \cdots$$
$$w_2\big(x_i^{k-1} \bigm| 1 \leqslant i \leqslant m\big) w_{r_k}\big(x_i^k \bigm| 1 \leqslant i \leqslant m\big) \cdots w_{r_n}\big(x_i^n \bigm| 1 \leqslant i \leqslant m\big)$$

hence by definition, $T \circ U$ is $(t; \underbrace{2, \ldots, 2}_{k-1 \text{ times}}, r_k, \ldots, r_n)$-summing and

$$\pi_{t;2,\ldots,2,r_k,\ldots,r_n}(T \circ U) \leqslant \pi_q(T)\delta_{p_1,\ldots,p_n}(U) B_{p_1} \cdots B_{p_{k-1}}.$$

We can prove now that in case $r \in [1,2]$ the answer to Question is **YES** for all natural numbers. We remark that in cases $r = 1$ and $n \geqslant 2$ this proof gives a different constant from that obtained in Theorem 4. □

Corollary 19.

(i) *Let n be a natural number, $r \in [1,\infty)$, $p, q \in (1,\infty)$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\pi_q \circ \delta_p \subset \pi_{r;\underbrace{2,\ldots,2}_{n-1 \text{ times}},r}.$$

In the previous formula in case $n = 1$, instead of $\pi_{r;\underbrace{2,\ldots,2}_{n-1 \text{ times}},r}$ we must write π_r.

(ii) *Let $r \in [1,\infty)$, $p, q \in (1,\infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $r \in [1,2]$, then for all natural numbers n we have*

$$\pi_q \circ \delta_p^n \subset \pi_r^n \quad \text{and} \quad \pi_q \circ \delta_p^n \subset \pi_2^n.$$

Proof. In case when $n = 1$, (i) and (ii) are the content of the Pietsch composition theorem and inclusion theorem for summing linear operators, $r \leqslant 2$.

Let $n \geqslant 2$. (i) By taking in Theorem 18 $k = n$, $r_n = r$, $p_n = p$, $q_n = q$ and $p_1 = \cdots = p_{n-1} = p$ we deduce

$$\pi_q \circ \delta_p \subset \pi_{t;\underbrace{2,\ldots,2}_{n-1 \text{ times}},r}$$

where $\frac{1}{t} = \frac{1}{p_n} + \frac{1}{q}$, which gives us the statement, because using the hypothesis we get $t = r$.

(ii) The first part of the statement follows from (i) since from $r \leqslant 2$, $w_2(\cdots) \leqslant w_r(\cdots)$ and thus $\pi_{r;\underbrace{2,\ldots,2}_{n-1 \text{ times}},r} \subset \pi_{r;\underbrace{r,\ldots,r}_{n-1 \text{ times}},r} = \pi_r$. For the second part of the statement, let $X_1 \times \cdots \times X_n \xrightarrow{U} Y \xrightarrow{T} Z$ be a diagram, where U is p-dominated and T is q-summing. Let $(x_i^j)_{1 \leqslant i \leqslant m} \subset X_j$ for $1 \leqslant j \leqslant n$. Let also $(\alpha_i)_{1 \leqslant i \leqslant m} \in l_s$ with $l_s(\alpha_i \mid 1 \leqslant i \leqslant m) \leqslant 1$, where $\frac{1}{r} = \frac{1}{2} + \frac{1}{s}$, $r \leqslant 2$. Then, by (i) we have

$$l_r\big(T\big(U\big(x_i^1, \ldots, x_i^{n-1}, \alpha_i x_i^n\big)\big) \bigm| 1 \leqslant i \leqslant m\big)$$
$$\leqslant \pi_q(T)\delta_p(U)[B_p]^{n-1} \cdot w_2\big(x_i^1 \bigm| 1 \leqslant i \leqslant m\big) \cdots w_2\big(x_i^{n-1} \bigm| 1 \leqslant i \leqslant m\big) w_r\big(\alpha_i x_i^n \bigm| 1 \leqslant i \leqslant m\big).$$

From here, since by Holder's inequality $w_r(\alpha_i x_i^n \mid 1 \leqslant i \leqslant m) \leqslant w_2(x_i^n \mid 1 \leqslant i \leqslant m) l_s(\alpha_i \mid 1 \leqslant i \leqslant m)$, we get

$$\left(\sum_{i=1}^{m} |\alpha_i|^r \left\| T\left(U\left(x_i^1, \ldots, x_i^{n-1}, x_i^n\right)\right)\right\|^r\right)^{\frac{1}{r}}$$
$$\leqslant \pi_q(T)\delta_p(U)[B_p]^{n-1} \cdot w_2\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_2\left(x_i^{n-1} \mid 1 \leqslant i \leqslant m\right) w_2\left(x_i^n \mid 1 \leqslant i \leqslant m\right).$$

Now taking supremum over $l_s(\alpha_i \mid 1 \leqslant i \leqslant m) \leqslant 1$ in the right member, we obtain

$$\left(\sum_{i=1}^{m} \left\| T\left(U\left(x_i^1, \ldots, x_i^{n-1}, x_i^n\right)\right)\right\|^2\right)^{\frac{1}{2}}$$
$$\leqslant \pi_q(T)\delta_p(U)[B_p]^{n-1} \cdot w_2\left(x_i^1 \mid 1 \leqslant i \leqslant m\right) \cdots w_2\left(x_i^{n-1} \mid 1 \leqslant i \leqslant m\right) w_2\left(x_i^n \mid 1 \leqslant i \leqslant m\right)$$

which means that $T \circ U$ is 2-summing and $\pi_2(T \circ U) \leqslant \pi_q(T)\delta_p(U)[B_p]^{n-1}$. □

Remark 20. We have include the second part in (ii), because in the multilinear setting, i.e. $n \geqslant 2$, there is no inclusion result for summing operators, see [13].

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